7.2 ANGULAR MOMENTUM

Objectives

- Define angular momentum.
- Explain the relationship between torque and rate of change of angular momentum.
- Define angular impulse.
- Explain the relationship between angular impulse and change in angular momentum.
- Explain the law of conservation of angular momentum.
- Solve problems using angular momentum, angular impulse, and conservation of angular momentum.

When we say an object is in *translation*, we imply that it moves in a straight line. When we say an object is in *rotation*, we imply that it spins or rotates around an axis. In Section 5.1 you learned the analogy between the equations for the kinetic energy of bodies in translation and rotation.

$$\mathrm{KE}_{\text{translation}} = \tfrac{1}{2}mv^2$$

$$\mathrm{KE}_{\text{rotation}} = \tfrac{1}{2}I\omega^2$$

In rotation, the moment of inertia I is analogous to mass, and angular velocity ω is analogous to velocity. The analogy is true also for momentum.

INTERNET *connection*

To find out more about angular momentum, follow the links at **www.learningincontext.com.**

Linear momentum tells you how difficult it is to stop an object moving in a straight line. *Angular momentum* tells you how difficult it is to stop a rotating object. The linear momentum of an object is the product of its mass and velocity. **Angular momentum** is the product of an object's moment of inertia and its angular velocity. We use the symbol L to represent angular momentum.

$$p = mv$$

$$L = I\omega$$

Remember that objects of equal mass but different shapes can have different moments of inertia. Formulas for calculating I were given in Section 5.1 for several common shapes. These formulas are repeated in Figure 7.7.

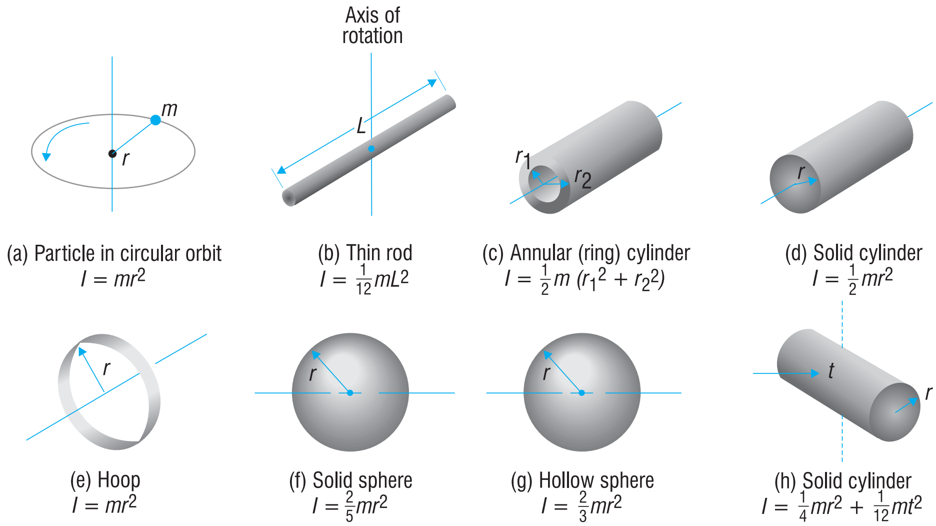


Figure 7.7
Moment of inertia formulas for different-shaped objects. In each formula, m is the mass of the object.

The SI and English units for moment of inertia, angular velocity, and angular momentum are listed in Table 7.2

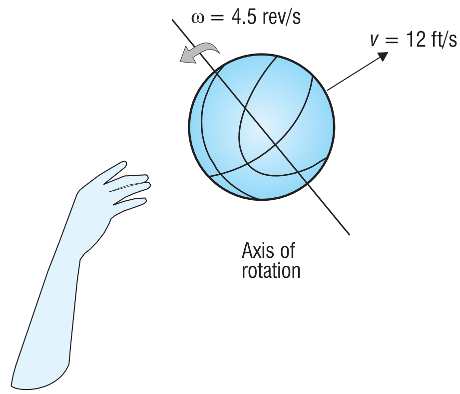
Table 7.2
Units Needed to Calculate Angular Momentum

$L = I\omega$	
SI Units	**English Units**
$I = \text{kg}\cdot\text{m}^2$	$I = \text{slug}\cdot\text{ft}^2$
$\omega = \text{rad/s}$	$\omega = \text{rad/s}$
$L = \text{kg}\cdot\text{m}^2/\text{s}$	$L = \text{slug}\cdot\text{ft}^2/\text{s}$

Angular momentum and angular velocity are vector quantities. But in this book we will work with only the magnitudes of these quantities.

Example 7.5 Linear and Angular Momenta of a Basketball

A basketball free-throw shooter gives the ball an initial velocity of 12 feet per second toward a point above the basket. (This is the velocity of the ball's center of mass.) The shooter also spins the ball at a rate of 4.5 revolutions per second. What are the magnitudes of the initial linear momentum and angular momentum of the basketball? A basketball weighs 21 ounces and has a radius of 0.40 foot.



Solution: Find the mass of the basketball in slugs. (1 lb = 16 oz)

$$m = \frac{(21\text{ oz})\left(\frac{1\text{ lb}}{16\text{ oz}}\right)}{32.2\text{ ft/s}^2} = 0.0408\text{ slug} \qquad \left[\text{slug} = \text{lb}\cdot\text{s}^2/\text{ft}\right]$$

A basketball is a hollow sphere. From Figure 7.7, the formula for the moment of inertia of a hollow sphere is:

$$I = \tfrac{2}{3}mr^2 = \tfrac{2}{3}(0.0408\text{ slug})(0.40\text{ ft})^2 = 4.35 \times 10^{-3}\text{ slug}\cdot\text{ft}^2$$

The angular speed must be in radians per second.

$$\omega = \left(4.5\,\frac{\text{rev}}{\text{s}}\right)\left(\frac{2\pi\text{ rad}}{\text{rev}}\right) = 28.3\text{ rad/s}$$

Now calculate the linear and angular momenta.

$$p = mv \qquad L = I\omega$$

$$p = (0.0408\text{ slug})(12\text{ ft/s}) \qquad L = (4.35 \times 10^{-3}\text{ slug}\cdot\text{ft}^2/\text{s})(28.3\text{ rad/s})$$

$$p = 0.49\text{ slug}\cdot\text{ft/s} \qquad L = 0.12\text{ slug}\cdot\text{ft}^2/\text{s}$$

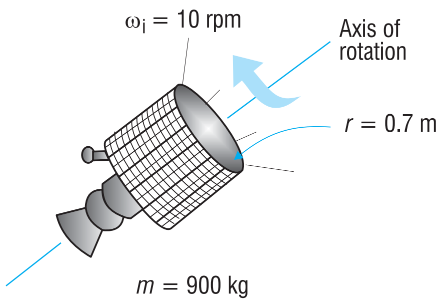
Newton's Second Law for Rotation

A net force is required to start (or stop) an object moving in translation. A net torque is required to start (or stop) an object moving around an axis of rotation. Torque in rotational motion is analogous to force in translational motion. Newton's second law relating force and linear momentum can be rewritten for torque and angular momentum.

Translation	***Rotation***
A net force exerted on an object equals the rate of change of the object's linear momentum.	*A net torque exerted on an object equals the rate of change of the object's angular momentum.*
$F = \dfrac{\Delta p}{\Delta t}$	$\tau = \dfrac{\Delta L}{\Delta t}$

Example 7.6 Force and Torque Required to Stop a Satellite

In an on-orbit repair mission, an astronaut volunteers to attempt to grab a satellite to stop its spin. The satellite's mass is 900 kg, and it is spinning at a rate of 10 rpm. The shape of the satellite can be modeled as a solid cylinder of radius 0.7 m.



The astronaut's boots will be firmly fixed to the space shuttle's robot arm. The shuttle will maintain a fixed attitude (will not rotate) in orbit by using its thrusters during the recovery. If the astronaut can grab a point on the outside surface of the satellite and hold on for 0.5 second, what force will be required to stop the spin?

Solution: Use the equation $\tau = \dfrac{\Delta L}{\Delta t}$ to calculate the torque required to stop the rotation. The torque is caused by a force F applied at a distance r from the center of rotation. When τ is known, you can calculate F from the equation $\tau = Fr$.

From Figure 7.7 for a solid cylinder, $I = \frac{1}{2}mr^2$.

$$I = \tfrac{1}{2}\,(900\text{ kg})(0.7\text{ m})^2 = 220.5\text{ kg}\cdot\text{m}^2$$

The initial angular velocity ω_i must be converted to radians per second.

$$\omega_i = \left(10\,\frac{\text{rev}}{\text{min}}\right)\left(\frac{1\text{ min}}{60\text{ s}}\right)\left(\frac{2\pi\text{ rad}}{\text{rev}}\right) = 1.05\text{ rad/s}$$

If the satellite stops, its final angular velocity ω_f is zero. The moment of inertia does not change. $(I_f = I_i = I)$

$$\tau = \frac{\Delta L}{\Delta t} = \frac{I\omega_f - I\omega_i}{\Delta t}$$

$$\tau = \frac{0 - (220.5\ \text{kg}\cdot\text{m}^2)(1.05\ \text{rad/s})}{0.5\ \text{s}}$$

$$\tau = -463\ \text{N}\cdot\text{m} \qquad [\text{kg}\cdot\text{m/s}^2 = \text{N}]$$

$$F = \frac{\tau}{r} = \frac{-463\ \text{N}\cdot\text{m}}{0.7\ \text{m}} = -661\ \text{N}$$

The negative sign means the torque and force must be applied in the direction opposite the satellite's motion.

The force required to stop the satellite in 0.5 s is 661 N.

Angular Impulse

To change the angular speed of a rotating object, such as the satellite in Example 7.6, a torque must be applied over a time interval. *Angular impulse* in rotational motion is analogous to linear impulse in translational motion. **Angular impulse** is the product of the torque τ and the time interval Δt over which the torque acts.

$$\text{Linear impulse} = F\Delta t \qquad \text{Angular impulse} = \tau\Delta t$$

A potter's wheel is a massive disk that rotates about an axis through its center. The potter uses angular impulse when she sets the wheel in motion, or accelerates a wheel that is already rotating. By pushing on the outside edge of the wheel, she applies a torque that equals the force times the lever arm. As shown in Figure 7.8, the lever arm is the distance from the applied force to the center of rotation.

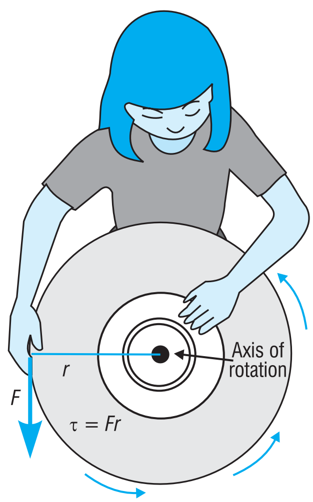


Figure 7.8
An angular impulse increases the momentum of a potter's wheel.

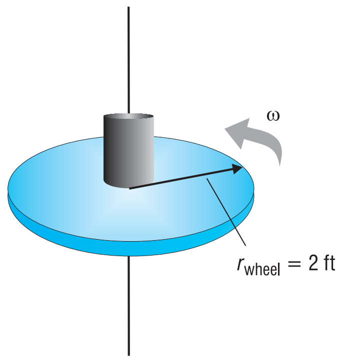
The greater the torque and the longer it is applied, the greater the angular impulse. The angular impulse determines the change in angular speed and momentum. This relationship is analogous to the relationship between linear impulse and momentum.

Translation	***Rotation***
Linear impulse = change in linear momentum	Angular impulse = change in angular momentum
$F\Delta t = \Delta p$	$\tau\Delta t = \Delta L$

Example 7.7 Torque on a Potter's Wheel

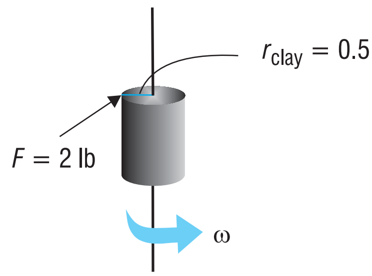
A 100-pound potter's wheel is 2 feet in radius. A cylinder of clay 6 inches in radius is fixed at the axis of rotation of the wheel. The rotational inertia of the wheel keeps the clay turning at a nearly constant rate while the potter forms the clay into a shape.

The wheel spins at an initial rate of 3.5 rev/s. The potter applies a frictional force of 2 lb on the outside surface of the clay cylinder. The force is applied over a time interval of 10 s. The clay weighs 3.8 lb and does not slip on the wheel as it is formed.



What torque is applied to the clay and wheel?

Solution: The torque is the applied force times the lever arm. In this case the lever arm is the radius of the clay cylinder. The force is frictional, so it opposes the rotation. (It slows the angular speed.) Therefore, it is a negative quantity and so is the torque.



$$\tau = Fr = -(2.0 \text{ lb})(0.5 \text{ ft})$$

$$\tau = -1.0 \text{ lb}\cdot\text{ft}$$

Example 7.8 Angular Speed of a Potter's Wheel

What is the angular speed of the potter's wheel in Example 7.7, after the torque is applied for 10 s? The radius of the clay does not change significantly over this time interval. Write the answer in rev/s.

Solution: Use the angular impulse-momentum equation. The combined moment of inertia I of the wheel and clay doesn't change. $(I_f = I_i = I)$

$$\tau\Delta t = \Delta L = I\omega_f - I\omega_i = I(\omega_f - \omega_i)$$

The moment of inertia of the combined shapes is the sum of the individual shapes. Both the wheel and clay are solid cylinders. From Figure 7.6, the moment of inertia of a solid cylinder is $\frac{1}{2}mr^2$.

$$I = I_{\text{wheel}} + I_{\text{clay}}$$

$$I = \tfrac{1}{2}m_{\text{wheel}}r_{\text{wheel}}{}^2 + \tfrac{1}{2}m_{\text{clay}}r_{\text{clay}}{}^2$$

$$I = \tfrac{1}{2}\left(\frac{100\text{ lb}}{32.2\text{ ft/s}^2}\right)(2\text{ ft})^2 + \tfrac{1}{2}\left(\frac{3.8\text{ lb}}{32.2\text{ ft/s}^2}\right)(0.5\text{ ft})^2$$

$$I = 6.23\text{ slug}\cdot\text{ft}^2 \qquad \left[\text{lb} = \text{slug}\cdot\text{ft/s}^2\right]$$

Angular speed must be in units of rad/s.

$$\omega_i = \left(3.5\frac{\text{rev}}{\text{s}}\right)\left(2\pi\frac{\text{rad}}{\text{rev}}\right) = 22.0\text{ rad/s}$$

Substitute the values into the angular impulse-momentum equation.

$$\tau\Delta t = I(\omega_f - \omega_i)$$

$$-(1.0\text{ lb}\cdot\text{ft})(10\text{ s}) = (6.23\text{ slug}\cdot\text{ft}^2)(\omega_f - 22.0\text{ rad/s})$$

$$-\frac{(1.0)(10)}{6.23}\frac{\text{rad}}{\text{s}} + 22.0\frac{\text{rad}}{\text{s}} = \omega_f \qquad \left[\text{slug} = \text{lb}\cdot\text{s}^2/\text{ft}\right]$$

$$\omega_f = 20.4\text{ rad/s}$$

Convert to rev/s.

$$\omega_f = \left(20.4\frac{\text{rad}}{\text{s}}\right)\left(\frac{1\text{ rev}}{2\pi\text{ rad}}\right) = 3.25\text{ rev/s}$$

The potter's wheel slows to 3.25 revolutions per second.

Conservation of Angular Momentum

In the last section, we derived the law of conservation of linear momentum by applying Newton's laws of motion to a closed system. The impulse applied to a closed system equals the change in the system's linear momentum. If no net force is applied to a system, the impulse is zero and there is no change in the system's linear momentum.

The same logic applies to a rotational system. Torque is analogous to force, and angular momentum is analogous to linear momentum.

Translation	***Rotation***
Linear impulse = change in linear momentum	Angular impulse = change in angular momentum
$F\Delta t = \Delta p$	$\tau\Delta t = \Delta L$
If $F = 0$, then $\Delta p = 0$ and $p_f = p_i$.	If $\tau = 0$, then $\Delta L = 0$ and $L_f = L_i$.

The result for rotational motion is called the **law of conservation of angular momentum**.

> *When no net external torque acts on a closed system, the total angular momentum of the system does not change.*

Skaters, gymnasts, dancers, and divers use conservation of angular momentum. For example, a diver changes from a pike position to an extended position at the end of a flip. This extension of mass away from the center of rotation increases the diver's moment of inertia I. There is no net external torque on the diver, angular momentum is conserved, and the product $I\omega$ is constant. When I is increased, ω is decreased. As the diver enters the water, her angular velocity is so small that it appears to be zero.

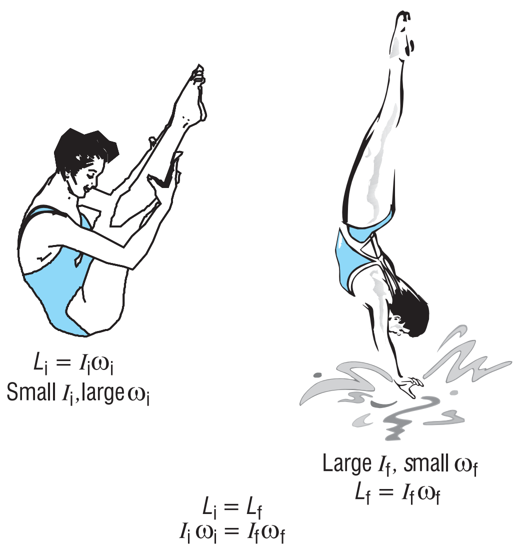
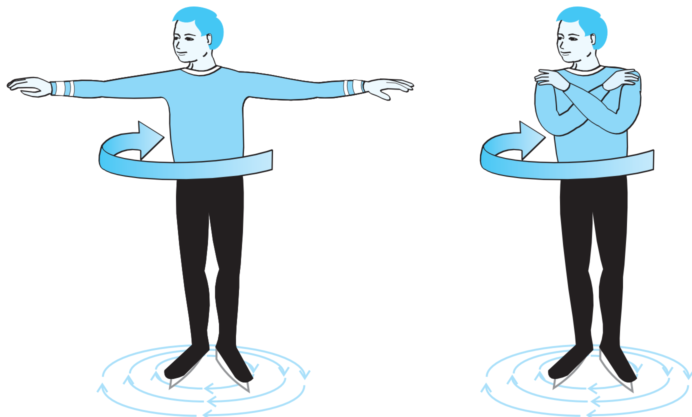


Figure 7.9
A diver's angular momentum is conserved. If $I\omega$ is constant, as I increases, ω decreases.

The diver in Figure 7.9 extends mass away from her center of rotation, and increases her moment of inertia. The skater in Example 7.9 below moves mass inward, toward his center of rotation. This decreases his moment of inertia. Will the skater's angular velocity increase or decrease?

Example 7.9 A Skater's Angular Momentum

A skater begins a spin with his arms outstretched. In this position, his moment of inertia is 4.6 kg•m² and he spins at a rate of 1.5 revolutions per second. The skater brings his arms close to his body to increase his spin rate. In this position, his moment of inertia is 1.4 kg•m². What is the skater's final angular speed?



Solution: The force and torque caused by friction between the skates and ice are very small. If this torque can be ignored, angular momentum is conserved.

$$L_f = L_i$$

$$I_f\omega_f = I_i\omega_i$$

$$\omega_f = \frac{I_i}{I_f}\omega_i = \left(\frac{4.6\ \text{kg}\cdot\text{m}^2}{1.4\ \text{kg}\cdot\text{m}^2}\right)(1.5\ \text{rev/s}) = 4.9\ \text{rev/s}$$

In this calculation, you do not need to convert angular speed to rad/s since the conversion factors would cancel.

The skater's final speed is 4.9 revolutions per second.

Example 7.10 Work Done by a Skater

How much work does the skater in Example 7.9 do when he pulls his arms in to his body?

Solution: From the work-energy theorem, the work done by the skater equals the change in kinetic energy.

$$W = \Delta \mathrm{KE} = \mathrm{KE_f} - \mathrm{KE_i}$$

The kinetic energy of an object in rotation is $\frac{1}{2} I\omega^2$.

$$\mathrm{KE_i} = \tfrac{1}{2} I_\mathrm{i} {\omega_\mathrm{i}}^2$$

$$= \tfrac{1}{2}(4.6\ \mathrm{kg \cdot m^2})\left(1.5 \frac{\mathrm{rev}}{\mathrm{s}} \bullet \frac{2\pi\ \mathrm{rad}}{\mathrm{rev}}\right)^2 = 204\ \mathrm{J}$$

$$\mathrm{KE_f} = \tfrac{1}{2} I_\mathrm{f} {\omega_\mathrm{f}}^2$$

$$= \tfrac{1}{2}(1.4\ \mathrm{kg \cdot m^2})\left(4.9 \frac{\mathrm{rev}}{\mathrm{s}} \bullet \frac{2\pi\ \mathrm{rad}}{\mathrm{rev}}\right)^2 = 664\ \mathrm{J}$$

$$W = 664\ \mathrm{J} - 204\ \mathrm{J} = 460\ \mathrm{J}$$

The skater does 460 joules of work.

Use dimensional analysis to show that the units in Example 7.10 are correct.

Summary

- Angular momentum is the product of an object's moment of inertia and its angular velocity. $L = I\omega$
- If a net torque is applied to an object, the torque equals the object's rate of change of angular momentum. $\tau = \dfrac{\Delta L}{\Delta t}$
- If a torque is applied to an object, the angular impulse is the torque times the time interval over which the torque is applied.
- When an angular impulse is applied to an object, the angular impulse equals the change in angular momentum. $\tau \Delta t = \Delta L$
- The angular momentum of a closed system is constant if no net external torque acts on the system.

Exercises

1. A spinning object has angular momentum equal to the product of the object's ____________ and ____________.
2. When a torque acts on a rotating object over an interval of time, the product of the torque and the time interval is called ____________.
3. Write an equation for Newton's second law for an object in translation, using force and momentum. Write an analogous equation for an object in rotation. What two pairs of variables in the equations are analogous?
4. Match the units on the left to the descriptions on the right.

____	(lb·ft)·s	a.	Angular momentum in SI units
____	(N·m)·s	b.	Angular momentum in English units
____	$kg \cdot m/s^2$	c.	Same as a slug
____	$slug \cdot ft^2/s$	d.	Angular impulse in SI units
____	$lb \cdot s^2/ft$	e.	Same as a newton
____	$kg \cdot m^2/s$	f.	Angular impulse in English units

5. Newton's second law for rotation can be written in the form $\tau = \Delta L/\Delta t$. Prove that the English units for the left side equal those for the right side.
6. The angular impulse-momentum equation can be written $\tau\Delta t = \Delta L$. Prove that the SI units for the left side equal those for the right side.
7. A pitcher throws a curve ball at an initial velocity of 75 mph (33.5 m/s) toward the plate. The pitcher also spins the ball at a rate of 30 revolutions per second. What are the magnitudes of the baseball's initial linear momentum and angular momentum? A baseball has a mass of 0.14 kg and a radius of 3.6 cm.
8. The mass of the Earth is 5.98×10^{24} kg and its radius is 6.37×10^6 m. What is the angular momentum of the Earth's spin about its polar axis?
9. A 300-kg flywheel is used to store energy in a punch press. The flywheel is in the shape of a solid cylinder of radius 0.8 m. When the operator engages the press, the flywheel slows from 250 rpm to 150 rpm in 6 seconds. What torque is applied to the flywheel?
10. A turbine rotates at an initial angular speed of 200 rad/s. A torque of 400 N·m causes the angular speed to double in 30 s. What is the moment of inertia of the turbine?

11. The moment of inertia of a complicated shape is difficult to calculate directly. Describe how you could measure the moment of inertia experimentally.

12. The 229,000-lb space shuttle orbiter is in a circular orbit 147 miles above the surface of the Earth. It completes an orbit every 89 minutes.

    (a) What is the angular speed of the orbiter in rpm? In rad/s?

    (b) What is the linear speed of the orbiter in mi/h? In ft/s? (The Earth's radius is 3960 miles.)

    (c) What is the moment of inertia of the orbiter?

    (d) What is the orbiter's angular momentum?

13. An ice skater begins a spin with her arms outstretched and then pulls them inward. Tell whether each of the following increases, decreases, or stays the same.

    (a) moment of inertia

    (b) spin rate

    (c) angular momentum

    (d) kinetic energy

14. Neutron stars are believed to be formed from the cores of larger stars that collapse under their own forces of gravity. The collapse can cause a supernova explosion. Prior to a collapse, suppose a large star has a core of radius 150,000 km. The core is a solid sphere that rotates once every 16 hours.

    (a) What is the angular speed of rotation in rad/s?

    (b) During the collapse, the outer layers of the star are blown off and the core shrinks to a radius of 15 km. There are no external forces to produce torques on the core, and the mass stays approximately the same. What is the new angular speed in rad/s? In rpm?